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Abstract

We examine maximin and minimax strategies for players in symmetric multi-players game with two strategic variables. We consider two patterns of game; the x-game in which strategic variables of players are x's, and the p-game in which strategic variables of players are p's. We will show that the maximin strategy and the minimax strategy in the x-game, and the maximin strategy and the minimax strategy in the p-game for the players are all equivalent. However, the maximin strategy for the players are not necessarily equivalent to their Nash equilibrium strategies in the x-game nor the p-game. But in a special case, where the objective function of one player is the opposite of the sum of the objective functions of other players, the maximin and the minimax strategies for the players constitute the Nash equilibrium both in the x-game and the p-game.

keywords: multi-players game, two strategic variables, maximin strategy, minimax strategy

JEL Classification: C72, D43.

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1 Introduction

We examine maximin and minimax strategies for players in symmetric multi-players game with two strategic variables. We consider two patterns of game; the x-game in which strategic variables of players are x's, and the p-game in which strategic variables of players are p's. The maximin strategy for a player is its strategy which maximizes its objective function that is minimized by a strategy of each rival player. The minimax strategy for a player is a strategy of each rival player which minimizes its objective function that is maximized by its strategy. These strategies are defined for any pair of two players. The objective functions of the players may be or may not be their absolute profits. We will show that the maximin strategy and the minimax strategy in the x-game, and the maximin strategy and the minimax strategy in the p-game for the players are all equivalent. However, the maximin strategy (or the minimax strategy) for the players are not necessarily equivalent to their Nash equilibrium strategies in the x-game nor the p-game. But in a special case, where the objective function of one player is the opposite of the sum of the objective functions of other players, the maximin strategy (or the minimax strategy) for the players constitute the Nash equilibrium both in the x-game and the p-game, and in the special case the Nash equilibrium in the x-game and that in the p-game are equivalent. This special case corresponds to relative profit maximization by firms in symmetric oligopoly with differentiated goods in which two strategic variables are the outputs and the prices.

In Section 5 we consider a mixed game in which some players choose x's and the other players choose p's as their strategic variables, and show that the maximin and minimax strategies for each player in the mixed game are equivalent to those in the x-game and the p-game.

2 The model

There are n players. Call each player Player i, $i \in \{1, 2, \dots, n\}$. The strategic variables of Player i are denoted by x_i and p_i. They are related by the following function.

$$p_i = f_i(x_1, x_2, \dots, x_n),\ i \in \{1, 2, \dots, n\}. \tag{1}$$

They are symmetric, continuous, differentiable and invertible. We consider symmetric equilibria. The inverses of them are written as

$$x_i = x_i(p_1, p_2, \dots, p_n),\ i \in \{1, 2, \dots, n\}.$$

Differentiating (1) with respect to p_i given p_j, $j \in \{1, 2, \dots, n\}$, $j \neq i$, yields

$$\frac{\partial f_i}{\partial x_i}\frac{dx_i}{dp_i} + \sum_{j=1, j\neq i}^{n} \frac{\partial f_i}{\partial x_j}\frac{dx_j}{dp_i} = 1.$$

$$\frac{\partial f_j}{\partial x_i}\frac{dx_i}{dp_i} + \frac{\partial f_j}{\partial x_j}\frac{dx_j}{dp_i} + \sum_{k=1, k\neq i,j}^{n} \frac{\partial f_j}{\partial x_k}\frac{dx_k}{dp_i} = 0,\ j \in \{1, 2, \dots, n\},\ j \neq i.$$

By symmetry of the model, since $\frac{\partial f_i}{\partial x_j}=\frac{\partial f_j}{\partial x_i}$ and $\frac{\partial f_j}{\partial x_j}=\frac{\partial f_i}{\partial x_i}$ at the equilibrium, they are rewritten as

$$\frac{\partial f_i}{\partial x_i}\frac{dx_i}{dp_i}+(n-1)\frac{\partial f_j}{\partial x_i}\frac{dx_j}{dp_i}=1.$$

$$\frac{\partial f_j}{\partial x_i}\frac{dx_i}{dp_i}+\left[\frac{\partial f_i}{\partial x_i}+(n-2)\frac{\partial f_j}{\partial x_i}\right]\frac{dx_j}{dp_i}=0.$$

From them we get

$$\frac{dx_i}{dp_i}=\frac{dx_j}{dp_j}=\frac{\frac{\partial f_i}{\partial x_i}+(n-2)\frac{\partial f_j}{\partial x_i}}{\left(\frac{\partial f_i}{\partial x_i}-\frac{\partial f_j}{\partial x_i}\right)\left[\frac{\partial f_i}{\partial x_i}+(n-1)\frac{\partial f_j}{\partial x_i}\right]} \tag{2}$$

and

$$\frac{dx_j}{dp_i}=\frac{dx_i}{dp_j}=-\frac{\frac{\partial f_j}{\partial x_i}}{\left(\frac{\partial f_i}{\partial x_i}-\frac{\partial f_j}{\partial x_i}\right)\left[\frac{\partial f_i}{\partial x_i}+(n-1)\frac{\partial f_j}{\partial x_i}\right]} \tag{3}$$

because $\frac{dx_i}{dp_j}=\frac{dx_j}{dp_i}$ and $\frac{dx_i}{dp_i}=\frac{dx_j}{dp_j}$ at the equilibrium. We assume

$$\frac{\partial f_i}{\partial x_i}\neq 0,\ \frac{\partial f_j}{\partial x_i}\neq 0,\ \frac{\partial f_i}{\partial x_i}-\frac{\partial f_j}{\partial x_i}\neq 0,\ \frac{\partial f_i}{\partial x_i}+(n-1)\frac{\partial f_j}{\partial x_i}\neq 0,\ \frac{\partial f_i}{\partial x_i}+(n-2)\frac{\partial f_j}{\partial x_i}\neq 0. \tag{4}$$

The objective function of Player $i,\ i\in\{1,2,\dots,n\}$ is

$$\pi_i(x_1,x_2,\dots,x_n).$$

It is continuous and differentiable. We consider two patterns of game, the x-game and the p-game. In the x-game strategic variables of players are x's, and in the p-game their strategic variables are p's. We do not consider simple maximization of their objective functions. Instead we investigate maximin strategies and minimax strategies for the players.

3 Maximin and minimax strategies

3.1 x-game

3.1.1 Maximin strategy

We pick up two players i and $j\neq i$. First consider the condition for minimization of π_i with respect to x_j, given x_i and x_k's, $k\in\{1,2,\dots,n\},\ k\neq i,j$. It is

$$\frac{\partial \pi_i}{\partial x_j}=0. \tag{5}$$

We assume that the second order condition is satisfied in each case. Depending on the value of x_i we get the value of x_j which satisfies (5). Denote it by $x_j(x_i)$. Differentiating (5) with respect to x_i given x_k's $k \in \{1, 2, \dots, n\},\ k \neq i, j$, we have

$$\frac{\partial^2 \pi_i}{\partial x_i^2} + \frac{\partial^2 \pi_i}{\partial x_i \partial x_j}\frac{dx_j(x_i)}{dx_i} = 0.$$

From this

$$\frac{dx_j(x_i)}{dx_i} = -\frac{\frac{\partial^2 \pi_i}{\partial x_i^2}}{\frac{\partial^2 \pi_i}{\partial x_j \partial x_i}}.$$

We assume that it is not zero. The maximin strategy for Player i to Player j is its strategy which maximizes π_i given $x_j(x_i)$ and x_k's $k \in \{1, 2, \dots, n\},\ k \neq i, j$. The condition for maximization of π_i is

$$\frac{\partial \pi_i}{\partial x_i} + \frac{\partial \pi_i}{\partial x_j}\frac{dx_j(x_i)}{dx_i} = 0.$$

By (5) it is reduced to

$$\frac{\partial \pi_i}{\partial x_i} = 0.$$

Thus, the conditions for the maximin strategy for Player i to Player j are

$$\frac{\partial \pi_i}{\partial x_i} = 0,\ \frac{\partial \pi_i}{\partial x_j} = 0. \tag{6}$$

(6) are the same for all pairs of $i \in \{1, 2, \dots, n\}$ and $j \neq i$.

3.1.2 Minimax strategy

Consider the condition for maximization of π_i with respect to x_i given x_j, and x_k's, $k \in \{1, 2, \dots, n\}, k \neq i, j$. It is

$$\frac{\partial \pi_i}{\partial x_i} = 0. \tag{7}$$

Depending on the value of x_j we get the value of x_i which satisfies (7). Denote it by $x_i(x_j)$. Differentiating (7) with respect to x_j given x_k's, $k \in \{1, 2, \dots, n\},\ k \neq i, j$.

$$\frac{\partial^2 \pi_i}{\partial x_i^2}\frac{dx_i}{dx_j} + \frac{\partial^2 \pi_i}{\partial x_j \partial x_i} = 0.$$

From it we obtain

$$\frac{dx_i(x_j)}{dx_j} = -\frac{\frac{\partial^2 \pi_i}{\partial x_j \partial x_i}}{\frac{\partial^2 \pi_i}{\partial x_i^2}}.$$

We assume that it is not zero. The minimax strategy for Player i to Player j is a strategy of Player j which minimizes π_i given $x_i(x_j)$ and x_k's ,$k \in \{1, 2, \dots, n\},\ k \neq i, j$. The condition for minimization of π_i is

$$\frac{\partial \pi_i}{\partial x_i}\frac{dx_i(x_j)}{dx_j} + \frac{\partial \pi_i}{\partial x_j} = 0.$$

By (7) it is reduced to

$$\frac{\partial \pi_i}{\partial x_j} = 0.$$

Thus, the conditions for the minimax strategy for Player i are

$$\frac{\partial \pi_i}{\partial x_i} = 0,\ \frac{\partial \pi_i}{\partial x_j} = 0.$$

These conditions are the same for all pairs of $i \in \{1, 2, \dots, n\}$ and $j \neq i$, and they are the same as conditions in (6).

3.2 p-game

We pick up two players i and $j \neq i$. The objective function of Player i, $i \in \{1, 2, \dots, n\}$, in the p-game is written as follows.

$$\pi_i(x_1(p_1, p_2, \dots, p_n), x_2(p_1, p_2, \dots, p_n), \dots, x_n(p_1, p_2, \dots, p_n)).$$

We can write it as

$$\pi_i(p_1, p_2, \dots, p_n),$$

because π_i is a function of $p_1, p_2, \dots, p_n$. Interchanging x_i, x_j and x_k by p_i, p_j and p_k in the arguments in the previous subsection, we can show that the conditions for the maximin strategy and the minimax strategy for Player i to Player j in the p-game are as follows.

$$\frac{\partial \pi_i}{\partial p_i} = 0,\ \frac{\partial \pi_i}{\partial p_j} = 0. \tag{8}$$

The conditions in (8) are the same for all pairs of $i \in \{1, 2, \dots, n\}$ and $j \neq i$. We can rewrite them as follows.

$$\frac{\partial \pi_i}{\partial p_i} = \frac{\partial \pi_i}{\partial x_i}\frac{dx_i}{dp_i} + (n-1)\frac{\partial \pi_i}{\partial x_j}\frac{dx_j}{dp_i} = 0,$$

$$\begin{aligned}\frac{\partial \pi_i}{\partial p_j} =& \frac{\partial \pi_i}{\partial x_i}\frac{dx_i}{dp_j} + \frac{\partial \pi_i}{\partial x_j}\frac{dx_j}{dp_j} + (n-2)\frac{\partial \pi_i}{\partial x_k}\frac{dx_k}{dp_j} \\ =& \frac{\partial \pi_i}{\partial x_i}\frac{dx_i}{dp_j} + \frac{\partial \pi_i}{\partial x_j}\left[\frac{dx_i}{dp_i} + (n-2)\frac{dx_i}{dp_j}\right] = 0,\ k \neq i, j,\end{aligned}$$

because $\frac{dx_j}{dp_j} = \frac{dx_i}{dp_i}$, $\frac{\partial \pi_i}{\partial x_k} = \frac{\partial \pi_i}{\partial x_j}$ and $\frac{dx_k}{dp_j} = \frac{dx_i}{dp_j}$ at the symmetric equilibrium. By (2) and (3) and the assumptions in (4), they are further rewritten as

$$\frac{\partial \pi_i}{\partial x_i}\left[\frac{\partial f_i}{\partial x_i} + (n-2)\frac{\partial f_j}{\partial x_i}\right] - (n-1)\frac{\partial \pi_i}{\partial x_j}\frac{\partial f_j}{\partial x_i} = 0,$$

$$\frac{\partial \pi_i}{\partial x_i}\frac{\partial f_j}{\partial x_i} - \frac{\partial \pi_i}{\partial x_j}\frac{\partial f_i}{\partial x_i} = 0.$$

Again by the assumptions in (4), we obtain

$$\frac{\partial \pi_i}{\partial x_i} = 0,\ \frac{\partial \pi_i}{\partial x_j} = 0.$$

They are the same as conditions in (6). We have proved the following proposition.

Proposition 1. *The maximin strategy and the minimax strategy in the x-game, and the maximin strategy and the minimax strategy in the p-game for the players are all equivalent.*

4 Special case

The results in the previous section do not imply that the maximin strategies (or the minimax strategies) for the players are equivalent to their Nash equilibrium strategies in the x-game nor the p-game. But in a special case the maximin strategies (or the minimax strategies) for the players constitute theNash equilibrium both in the x-game and the p-game.

The conditions for the maximin strategy and the minimax strategy for Player i to Player j are

$$\frac{\partial \pi_i}{\partial x_i} = 0,\ \frac{\partial \pi_i}{\partial x_j} = 0,\ j \neq i,\ i \in \{1, 2, \dots, n\}. \tag{6}$$

The conditions for Nash equilibrium in the x-game for Players i and j are

$$\frac{\partial \pi_i}{\partial x_i} = 0,\ \frac{\partial \pi_j}{\partial x_j} = 0,\ j \neq i,\ i \in \{1, 2, \dots, n\}. \tag{9}$$

(6) and (9) are not necessarily equivalent. The conditions for Nash equilibrium in the p-game are

$$\frac{\partial \pi_i}{\partial p_i} = 0,\ \frac{\partial \pi_j}{\partial p_j} = 0,\ j \neq i,\ i \in \{1, 2, \dots, n\}. \tag{10}$$

(8) and (10) are not necessarily equivalent. However, in a special case those conditions are all equivalent. We assume

$$\pi_i = -\sum_{j=1, j\neq i}^{n} \pi_j, \text{ or } \pi_i + \sum_{j=1, j\neq i}^{n} \pi_j = 0. \tag{11}$$

By symmetry of the game

$$\pi_i = -(n-1)\pi_j,\ j \neq i.$$

Then, (9) is rewritten as

$$\frac{\partial \pi_i}{\partial x_i} = 0,\ \frac{\partial \pi_i}{\partial x_j} = 0,\ j \neq i,\ i \in \{1, 2, \dots, n\}. \tag{12}$$

(12) and (6) are equivalent. Therefore, the maximin strategies and the minimax strategies for the players in the x-game constitute a Nash equilibrium of the x-game. $\frac{\partial \pi_j}{\partial x_j} = 0$ in (9) means

maximization of π_j with respect to x_j. On the other hand, $\frac{\partial \pi_i}{\partial x_j} = 0$ in (12) and (6) means minimization of π_i with respect to x_j.

Similarly, (10) is rewritten as

$$\frac{\partial \pi_i}{\partial p_i} = 0,\ \frac{\partial \pi_i}{\partial p_j} = 0,\ j \neq i,\ i \in \{1, 2, \ldots, n\}. \tag{13}$$

(13) and (8) are equivalent. Therefore, the maximin strategies and the minimax strategies for the players in the p-game constitute a Nash equilibrium of the p-game. Since the maximin strategies and the minimax strategies in the x-game and those in the p-game are equivalent, the Nash equilibrium of the x-game and that of the p-game are equivalent.

Summarizing the results, we get the following proposition.

Proposition 2. *In the special case in which (11) is satisfied: The maximin strategies and the minimax strategies for the players constitute the Nash equilibrium both in the x-game and the p-game.*

This special case corresponds to relative profit maximization by firms in symmetric oligopoly with differentiated goods in which two strategic variables are the outputs and the prices[1]. Let $\bar{\pi}_i$ be the absolute profit of Firm i, $i \in \{1, 2, \ldots, n\}$, and denote its relative profit by π_i. Then,

$$\pi_i = \bar{\pi}_i - \frac{1}{n-1} \sum_{j=1, j \neq i}^{n} \bar{\pi}_j,\ i \in \{1, 2, \ldots, n\}.$$

We have

$$\sum_{i=1}^{n} \pi_i = \sum_{i=1}^{n} \bar{\pi}_i - \sum_{i=1}^{n} \bar{\pi}_i = 0.$$

By symmetry of the oligopoly

$$\pi_i = -(n-1)\pi_j.$$

5 Mixed game

Suppose that the first m players choose p's and the remaining $n - m$ players choose x's as their strategic variables. We assume $1 \leq m \leq n - 1$. Differentiating (1) with respect to p_i, $i = 1, \ldots, m$, given p_k, $k = 1, \ldots, m, k \neq i$, and x_j, $j = m + 1, \ldots, n$,

$$\frac{\partial f_i}{\partial x_i}\frac{dx_i}{dp_i} + (m-1)\frac{\partial f_i}{\partial x_k}\frac{dx_k}{dp_i} = 1,\ k \in \{1, \ldots, m\},\ k \neq i,$$

$$\frac{\partial f_k}{\partial x_k}\frac{dx_k}{dp_i} + \frac{\partial f_k}{\partial x_i}\frac{dx_i}{dp_i} + (m-2)\frac{\partial f_k}{\partial x_{k'}}\frac{dx_{k'}}{dp_i} = 0,\ k \in \{1, \ldots, m\},\ k \neq i,\ k' \neq i, k,$$

[1] About relative profit maximization under imperfect competition, please see Matsumura, Matsushima and Cato (2013), Satoh and Tanaka (2013), Satoh and Tanaka (2014a), Satoh and Tanaka (2014b), Tanaka (2013a), Tanaka (2013b) and Vega-Redondo (1997).

At the equilibrium we assume $\frac{dx_{k'}}{dx_i}=\frac{dx_k}{dx_i}$, $\frac{\partial f_k}{\partial x_k}=\frac{\partial f_i}{\partial x_i}$, $\frac{\partial f_i}{\partial x_k}=\frac{\partial f_k}{\partial x_{k'}}=\frac{\partial f_k}{\partial x_i}$. Then, they are rewritten as

$$\frac{\partial f_i}{\partial x_i}\frac{dx_i}{dp_i}+(m-1)\frac{\partial f_k}{\partial x_i}\frac{dx_k}{dp_i}=1,$$

$$\frac{\partial f_k}{\partial x_i}\frac{dx_i}{dp_i}+\left[\frac{\partial f_i}{\partial x_i}+(m-2)\frac{\partial f_k}{\partial x_i}\right]\frac{dx_k}{dp_i}=0,$$

From them we get

$$\frac{dx_i}{dp_i}=\frac{\frac{\partial f_i}{\partial x_i}+(m-2)\frac{\partial f_k}{\partial x_i}}{\left(\frac{\partial f_i}{\partial x_i}-\frac{\partial f_k}{\partial x_i}\right)\left[\frac{\partial f_i}{\partial x_i}+(m-1)\frac{\partial f_k}{\partial x_i}\right]},$$

$$\frac{dx_k}{dp_i}=-\frac{\frac{\partial f_k}{\partial x_i}}{\left(\frac{\partial f_i}{\partial x_i}-\frac{\partial f_k}{\partial x_i}\right)\left[\frac{\partial f_i}{\partial x_i}+(m-1)\frac{\partial f_k}{\partial x_i}\right]}.$$

We assume

$$\frac{dx_i}{dp_i}-\frac{dx_k}{dp_i}=\frac{\frac{\partial f_i}{\partial x_i}+(m-1)\frac{\partial f_k}{\partial x_i}}{\left(\frac{\partial f_i}{\partial x_i}-\frac{\partial f_k}{\partial x_i}\right)\left[\frac{\partial f_i}{\partial x_i}+(m-1)\frac{\partial f_k}{\partial x_i}\right]}\neq 0, \tag{14}$$

$$\frac{dx_i}{dp_i}+(m-1)\frac{dx_k}{dp_i}=\frac{\frac{\partial f_i}{\partial x_i}-\frac{\partial f_k}{\partial x_i}}{\left(\frac{\partial f_i}{\partial x_i}-\frac{\partial f_k}{\partial x_i}\right)\left[\frac{\partial f_i}{\partial x_i}+(m-1)\frac{\partial f_k}{\partial x_i}\right]}\neq 0. \tag{15}$$

Differentiating (1) with respect to $x_j,\ j=m+1,\dots,n$, given $p_i,\ i=1,\dots,m$, and $x_l,\ l=m+1,\dots,n,\ l\neq j$,

$$\frac{\partial f_i}{\partial x_i}\frac{dx_i}{dx_j}+(m-1)\frac{\partial f_i}{\partial x_k}\frac{dx_k}{dx_j}+\frac{\partial f_i}{\partial x_j}=0,\ i\in\{1,\dots,m\},\ k\neq i.$$

At the equilibrium we assume $\frac{dx_k}{dx_j}=\frac{dx_i}{dx_j}$, $\frac{\partial f_i}{\partial x_k}=\frac{\partial f_k}{\partial x_i}$. Then, it is rewritten as

$$\left[\frac{\partial f_i}{\partial x_i}+(m-1)\frac{\partial f_k}{\partial x_i}\right]\frac{dx_i}{dx_j}+\frac{\partial f_i}{\partial x_j}=0,$$

This means

$$\frac{dx_i}{dx_j}=-\frac{\frac{\partial f_i}{\partial x_j}}{\frac{\partial f_i}{\partial x_i}+(m-1)\frac{\partial f_k}{\partial x_i}},$$

We assume $\frac{dx_i}{dx_j}\neq 0$.

We write the objective functions as follows.

$$\varphi_i(p_1,\dots,p_m,x_{m+1},\dots,x_n)=\pi_i(x_1(p_1,\dots,p_n),\dots,x_m(p_1,\dots,p_n),x_{m+1},\dots,x_n),$$
$$i\in\{1,\dots,n\}.$$

Then,

$$\frac{\partial \varphi_i}{\partial p_i} = \frac{\partial \pi_i}{\partial x_i}\frac{dx_i}{dp_i} + (m-1)\frac{\partial \pi_i}{\partial x_k}\frac{dx_k}{dp_i},$$

$$\frac{\partial \varphi_i}{\partial p_k} = \frac{\partial \pi_i}{\partial x_i}\frac{dx_i}{dp_k} + \frac{\partial \pi_i}{\partial x_k}\frac{dx_k}{dp_k} + (m-2)\frac{\partial \pi_i}{\partial x_{k'}}\frac{dx_{k'}}{dp_k},$$

$$\frac{\partial \varphi_i}{\partial x_j} = \frac{\partial \pi_i}{\partial x_j} + \frac{\partial \pi_i}{\partial x_i}\frac{dx_i}{dx_j} + (m-1)\frac{\partial \pi_i}{\partial x_k}\frac{dx_k}{dx_j},$$

$$\frac{\partial \varphi_j}{\partial x_j} = \frac{\partial \pi_j}{\partial x_j} + m\frac{\partial \pi_j}{\partial x_i}\frac{dx_i}{dx_j},$$

$$\frac{\partial \varphi_j}{\partial x_l} = \frac{\partial \pi_j}{\partial x_l} + m\frac{\partial \pi_j}{\partial x_i}\frac{dx_i}{dx_l},$$

$$\frac{\partial \varphi_j}{\partial p_i} = \frac{\partial \pi_j}{\partial x_i}\frac{dx_i}{dp_i} + (m-1)\frac{\partial \pi_j}{\partial x_k}\frac{dx_k}{dp_i},$$

where $i \in \{1,\dots,m\}$, $k \in \{1,\dots,m\}$, $k \neq i$, $k' \neq i,k$, $j \in \{m+1,\dots,n\}$, $l \in \{m+1,\dots,n\}$, $l \neq j$. At the equilibrium $\frac{dx_k}{dp_k} = \frac{dx_i}{dp_i}$, $\frac{dx_{k'}}{dp_k} = \frac{dx_i}{dp_k} = \frac{dx_k}{dp_i}$, $\frac{\partial \pi_i}{\partial x_{k'}} = \frac{\partial \pi_i}{\partial x_k}$, $\frac{dx_i}{dx_l} = \frac{dx_i}{dx_j}$ and $\frac{\partial \pi_j}{\partial x_k} = \frac{\partial \pi_j}{\partial x_i}$. Then, they are rewritten as

$$\frac{\partial \varphi_i}{\partial p_i} = \frac{\partial \pi_i}{\partial x_i}\frac{dx_i}{dp_i} + (m-1)\frac{\partial \pi_i}{\partial x_k}\frac{dx_k}{dp_i},$$

$$\frac{\partial \varphi_i}{\partial p_k} = \frac{\partial \pi_i}{\partial x_i}\frac{dx_k}{dp_i} + \frac{\partial \pi_i}{\partial x_k}\left[\frac{dx_i}{dp_i} + (m-2)\frac{dx_k}{dp_i}\right],$$

$$\frac{\partial \varphi_i}{\partial x_j} = \frac{\partial \pi_i}{\partial x_i}\frac{dx_i}{dx_j} + \frac{\partial \pi_i}{\partial x_j} + (m-1)\frac{\partial \pi_i}{\partial x_k}\frac{dx_i}{dx_j}.$$

$$\frac{\partial \varphi_j}{\partial x_j} = \frac{\partial \pi_j}{\partial x_j} + m\frac{\partial \pi_j}{\partial x_i}\frac{dx_i}{dx_j},$$

$$\frac{\partial \varphi_j}{\partial x_l} = \frac{\partial \pi_j}{\partial x_l} + m\frac{\partial \pi_j}{\partial x_i}\frac{dx_i}{dx_l},$$

$$\frac{\partial \varphi_j}{\partial p_i} = \frac{\partial \pi_j}{\partial x_i}\left[\frac{dx_i}{dp_i} + (m-1)\frac{dx_k}{dp_i}\right].$$

By similar arguments to those in the previous sections, we obtain the conditions for the maximin and minimax strategies for Player i, $i \in \{1,\dots,m\}$, to Player j with the condition for the maximin and minimax strategies for Player i to Player k as follows;

$$\frac{\partial \varphi_i}{\partial p_i} = 0,\ \frac{\partial \varphi_i}{\partial p_k} = 0,\ \frac{\partial \varphi_i}{\partial x_j} = 0,\ i,k \in \{1,\dots,m\},\ k \neq i,\ j \in \{m+1,\dots,n\}. \tag{16}$$

From these conditions we obtain

$$\frac{\partial \pi_i}{\partial x_i}\frac{dx_i}{dp_i} + (m-1)\frac{\partial \pi_i}{\partial x_k}\frac{dx_k}{dp_i} = 0, \tag{17}$$

$$\frac{\partial \pi_i}{\partial x_i}\frac{dx_k}{dp_i} + \frac{\partial \pi_i}{\partial x_k}\left[\frac{dx_i}{dp_i} + (m-2)\frac{dx_k}{dp_i}\right] = 0, \tag{18}$$

$$\frac{\partial \pi_i}{\partial x_i}\frac{dx_i}{dx_j} + \frac{\partial \pi_i}{\partial x_j} + (m-1)\frac{\partial \pi_i}{\partial x_k}\frac{dx_i}{dx_j} = 0. \tag{19}$$

By (14) and (15), (17) and (18) imply

$$\frac{\partial \pi_i}{\partial x_i} = 0,\ \frac{\partial \pi_i}{\partial x_k} = 0,\ i \in \{1, \dots, m\},\ k \in \{1, \dots, m\},\ k \neq i. \tag{20}$$

From (19) we get

$$\frac{\partial \pi_i}{\partial x_j} = 0,\ i \in \{1, \dots, m\},\ j \in \{m+1, \dots, n\}. \tag{21}$$

(20) and (21) are the same as the conditions in (6) for Player i, $i \in \{1, \dots, m\}$.

The conditions for the maximin and minimax strategies for Player j, $j \in \{m+1, \dots, n\}$, to Player i with the condition for the maximin and minimax strategies for Player j to Player l are

$$\frac{\partial \varphi_j}{\partial x_j} = 0,\ \frac{\partial \varphi_j}{\partial x_l} = 0,\ \frac{\partial \varphi_j}{\partial p_i} = 0,\ j \in \{m+1, \dots, n\}, l \in \{m+1, \dots, n\},\ l \neq j,\ i \in \{1, \dots, m\}.$$

From them we obtain

$$\frac{\partial \pi_j}{\partial x_j} + m\frac{\partial \pi_j}{\partial x_i}\frac{dx_i}{dx_j} = 0, \tag{22}$$

$$\frac{\partial \pi_j}{\partial x_l} + m\frac{\partial \pi_j}{\partial x_i}\frac{dx_i}{dx_l} = 0, \tag{23}$$

$$\frac{\partial \pi_j}{\partial x_i}\left[\frac{dx_i}{dp_i} + (m-1)\frac{dx_k}{dp_i}\right] = 0. \tag{24}$$

From (15) and (24) we get

$$\frac{\partial \pi_j}{\partial x_i} = 0. \tag{25}$$

Then, by (22) and (23), we obtain

$$\frac{\partial \pi_j}{\partial x_j} = 0,\ \frac{\partial \pi_j}{\partial x_l} = 0. \tag{26}$$

(25) and (26) are the same as the conditions in (6) for Player j, $j \in \{m+1, \dots, n\}$.

Therefore, the conditions for the maximin and minimax strategies in the mixed game are equivalent to the conditions in the x-game.

6 Concluding Remark

We have analyzed maximin and minimax strategies in symmetric multi-players game with two strategic variables. We assumed differentiability of objective functions of players. In the future research we want to extend the arguments of this paper to a case where objective functions of players are not assumed to be differentiable[2].

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[2] One attempt along this line is Satoh and Tanaka (2017).